Nonlinear geostrophic adjustment in the presence of a boundary

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The process of nonlinear geostrophic adjustment in the presence of a boundary (i.e. in a half-plane bounded by a rigid wall) is examined in the framework of a rotating shallow water model, using an asymptotic multiple-time-scale theory based on the assumed smallness of the Rossby number ε. The spatial scale is of the order of the Rossby scale. Different initial states are considered: periodic, 'step'-like, and localized. In all cases the initial perturbation is split in a unique way into slow and fast components evolving with characteristic time scales f^{-1} and $(\varepsilon f)^{-1}$, respectively. The slow component is not influenced by the fast one, at least for times $t \leqslant (f\varepsilon)^{-1}$, and remains close to geostrophic balance. The fast component consists mainly of linear inertia–gravity waves rapidly propagating outward from the initial disturbance and Kelvin waves confined near the boundary.

The theory provides simple formulae allowing us to construct the initial profile of the Kelvin wave, given arbitrary initial conditions. With increasing time, the Kelvin wave profile gradually distorts due to nonlinear-wave self-interaction, the distortion being described by the equation of a simple wave. The presence of Kelvin waves does not prevent the fast–slow splitting, in spite of the fact that the frequency gap between the Kelvin waves and slow motion is absent. The possibility of such splitting is explained by the special structure of the Kelvin waves in each case considered.

The slow motion on time scales $t \leqslant (\varepsilon f)^{-1}$ is governed by the well-known quasi-geostrophic potential vorticity equation for the elevation. The theory provides an algorithm to determine initial slow and fast fields, and the boundary conditions to any order in ε. For the periodic and step-like initial conditions, the slow component behaves in the usual way, conserving mass, energy and enstrophy. In the case of a localized initial disturbance the total mass of the lowest-order slow component is not conserved, and conservation of the total mass is provided by the first-order slow correction and the Kelvin wave.

On longer time scales $t \leqslant (\varepsilon^2 f)^{-1}$ the slow motion obeys the so-called modified quasi-geostrophic potential vorticity (QGPV) equation. The theory provides initial and boundary conditions for this equation. This modified equation coincides exactly with the 'improved' QGPV equation, derived by Reznik, Zeitlin & Ben Jelloul (2001), in the step-like and localized cases. In the periodic case this equation contains an additional term due to the Kelvin-wave self-interaction, this term depending on the initial Kelvin wave profile.

1. Introduction

Geostrophic adjustment in a rotating fluid is the tendency for the large-scale part of the initial perturbation to reach a state of geostrophic equilibrium; at the same time the remaining part, consisting of rapidly propagating inertia–gravity (IG) waves gradually decays at a fixed space point with increasing time. The study of this process (playing a very important role in atmospheric and oceanic dynamics) started with the pioneering work by Rossby (1938). A thorough linear analysis of geostrophic adjustment has been given in a number of well-known works (e.g. Obukhov 1949; Monin & Obukhov 1958; Gill 1976). The lowest-order nonlinear corrections were discussed in the review by Blumen (1972) (where an extensive bibliography of the early works can be found) and more recently by Dewar & Killworth (1995).

The problem of nonlinear geostrophic adjustment is closely related to the balanced models which are used to describe the slow (balanced) part of the motion (e.g. Warn *et al.* 1995; Medvedev 1997). The fundamental question concerns the possibility of splitting an arbitrary motion into a slow (balanced) and a fast component in such a way that the slow component is not influenced by the fast one for long enough times. Note that the splitting is a more general concept than adjustment, since splitting is possible when the fast component consists of 'non-radiating' oscillations and permanently co-exists with the slow component, as in the cases of periodic motion or frontal dynamics. Periodic (in both horizontal directions) motion was examined by Embid & Majda (1996) and Babin, Mahalov & Nikolaenko (1998*a, b*) in the framework of the barotropic rotating shallow water (RSW) model when the relative elevation and the Rossby number ε are small. It was shown that the resulting field is split in a unique way into slow and fast components evolving with characteristic time scales f^{-1} and $(\varepsilon f)^{-1}$, respectively, where f is the Coriolis parameter. The slow component is not influenced by the fast one and remains close to geostrophic balance, being governed by the well-known quasi-geostrophic (QG) dynamics. The fast component is a superposition of IG waves with amplitudes slowly changing in time due to nonlinear interactions between the waves and slow motion (serving here as a catalyst).

These results were generalized by Reznik, Zeitlin & Ben Jelloul (2001, hereafter referred to as RZB) to the case of an arbitrary localized perturbation on an unbounded plane. The scenario of adjustment depends on the characteristic scale and/or initial relative elevation of the free surface. For small relative elevations the slow motion obeys the well-known QG equation for times $t \leqslant (f\varepsilon)^{-1}$ while modifications to this equation for longer times $t \leqslant (f\varepsilon^2)^{-1}$ are found. The fast component consists mainly of linear IG waves rapidly propagating outward from the initial perturbation; the nonlinear interaction between these waves and the slow component is of no importance, unlike the periodic case. For large relative elevations (the frontal dynamics regime) the slow field is governed by the frontal geostrophic dynamics equation. The fast component in this case is a spatially localized packet of inertial oscillations evolving on the background of the slow component of the flow and experiencing slow modulations obeying a Schrödinger-type equation with coefficients depending on the slow motion.

The physical reasons for the slow–fast splitting are the Lagrangian conservation of potential vorticity, the fact that IG waves do not carry the potential vorticity, and the gap in the spectrum of the small perturbation in the RSW model due to rotation which, in particular, blocks the Lighthill radiation of IG waves.

The question addressed in this paper is how these results are modified in the
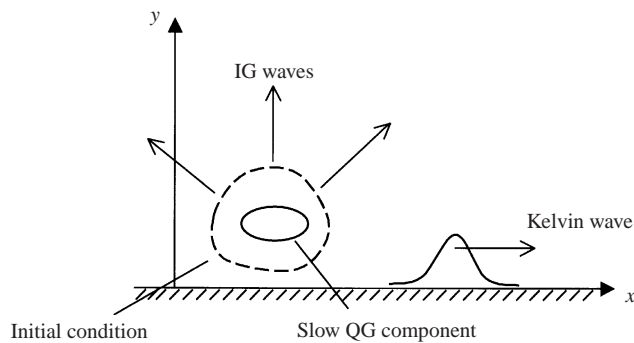
FIGURE 1. Schematic representation of geostrophic adjustment in a half-plane bounded by a rigid wall at $y = 0$.

presence of a horizontal boundary (see figure 1). This question is clearly of special interest for oceanographers. An essential new factor absent in unbounded fluid is the presence of Kelvin waves propagating along the boundaries in the rotating fluid. Gill (1976) examined the linear geostrophic adjustment of the initial discontinuity in free-surface height in a channel and revealed that the Kelvin waves play a key role in establishing the boundary conditions for the resulting geostrophic mode. Hermann, Rhines & Johnson (1989) investigated further the slow development of this mode, presuming its evolution to be governed by the usual equation for QG potential vorticity. Their numerical analysis using primitive RSW equations demonstrated the suitability of this approach.

In the present work we investigate analytically the influence of the boundaries on the process of nonlinear geostrophic adjustment. Note that the Kelvin waves can be of even more importance for the nonlinear adjustment than for the linear one, since there is no spectral gap between the slow mode and Kelvin waves: these waves can possess arbitrarily small frequencies. This means that the Lighthill radiation of the Kelvin waves is possible, at least, in principle. For simplicity we use again the barotropic RSW model and consider the motion on a half-plane bounded by a rigid wall. Initial fields can have an arbitrary form but their typical scale should not exceed the Rossby scale and the Rossby number is assumed to be small. The asymptotic multiple-time-scale procedure developed in RZB is applied for analysis.

The paper is organized as follows. In §2 the model is formulated. In §3 we examine the lowest-order solution for various initial conditions; the solution obtained describes the *linear* geostrophic adjustment of an arbitrary initial field on the half-plane. Nonlinear dynamics of the lowest-order slow motion is analysed in §4. The first-order solution is discussed in §5; we demonstrate that slow–fast splitting is possible at higher orders in Rossby number, at least up to terms $O(\varepsilon^2)$. A modified quasi-geostrophic potential vorticity equation describing the slow component on time scales longer than typical geostrophic time scale, for $t \leqslant (f\varepsilon^2)^{-1}$, is derived in §6. A summary of our results is presented in §7. The asymptotic behaviour of the lowest- and first-order fields is considered in Appendices A and B, which are available on request from the authors, or the Journal of Fluid Mechanics Editorial Office, Cambridge.

2. Model

The RSW model consists of the horizontal momentum and mass conservation equations for the thin free-surface layer under the influence of the Coriolis force and

gravity on the rotating plane (x, y):

$$\frac{\partial u}{\partial t} + \varepsilon \left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} \right) - v = -\frac{\partial h}{\partial x}, \tag{2.1a}$$

$$\frac{\partial v}{\partial t} + \varepsilon \left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} \right) + u = -\frac{\partial h}{\partial y}, \tag{2.1b}$$

$$\frac{\partial h}{\partial t} + (1 + \varepsilon h) \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + \varepsilon \left(u\frac{\partial h}{\partial x} + v\frac{\partial h}{\partial y} \right) = 0, \tag{2.1c}$$

where $\boldsymbol{v} = (u(x, y, t), v(x, y, t))$ is the two-dimensional velocity field and $H = H_0$ $(1 + \varepsilon h(x, y, t))$ is the free-surface elevation with the rest state corresponding to the constant H_0. Equations (2.1) are written in non-dimensional form, the Rossby scale $R_d = \sqrt{gH_0}/f$ and the reciprocal Coriolis parameter f^{-1} being chosen as the space and the time scales, respectively; g is the acceleration due to gravity, the parameter $\varepsilon = U/fR_d$ is the Rossby number, and U is the horizontal velocity scale.

The fields u, v, h are known at the initial moment,

$$u = u_I(x, y), \quad v = v_I(x, y), \quad h = h_I(x, y) \quad \text{at} \quad t = 0, \tag{2.2}$$

and obey the no-flux boundary condition on the rigid wall $y = 0$,

$$v = 0 \quad \text{at} \quad y = 0. \tag{2.3}$$

The initial fields (2.2) are not entirely arbitrary and satisfy the following conditions at the wall $y = 0$:

$$v_I = 0, \quad u_I = -\frac{\partial h_I}{\partial y} \quad \text{at} \quad y = 0, \tag{2.4a, b}$$

to be consistent with equations (2.1b), (2.3).

The vorticity equation follows from (2.1),

$$\frac{\partial(\zeta - h)}{\partial t} + \varepsilon \left\{ \frac{\partial[u(\zeta - h)]}{\partial x} + \frac{\partial[v(\zeta - h)]}{\partial y} \right\} = 0, \tag{2.5}$$

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}. \tag{2.6}$$

It is seen from (2.1), (2.5) that the fields u, v, h on the one hand and the vorticity field $\zeta - h$ on the other hand, are characterized, generally, by different time scales. The typical time scale of u, v, h is equal to the inertial time $T_i = f^{-1}$ and $\zeta - h$ changes in time at the advective time scale $T_a = R_d/U$. The Rossby number ε is the ratio between these scales,

$$\frac{T_i}{T_a} = \frac{U}{fR_d} = \varepsilon. \tag{2.7}$$

In what follows we assume the advective time scale to be much larger than the inertial one, which means that the Rossby number is small,

$$\varepsilon \ll 1. \tag{2.8}$$

Both the fast changes due to IG wave activity and the slow changes of the vorticity $\zeta - h$ are present in the evolution of the initial field (2.2) and the problem of *nonlinear adjustment* is to determine their mutual influence.

Three types of the initial conditions (u_I, v_I, h_I) will be considered:

(i) periodic (in x) boundary conditions,

$$(u_I, v_I, h_I) = \sum_{m=-\infty}^{m=\infty} (u_{Im}, v_{Im}, h_{Im}) \, e^{imx}, \tag{2.9a}$$

where the functions u_{Im}, v_{Im}, h_{Im} depend on y only;

(ii) a zonal 'step',

$$(u_I, v_I, h_I) \rightarrow (u_I^{(\pm)}(y), 0, h_I^{(\pm)}(y)), \quad x \rightarrow \pm\infty; \tag{2.9b}$$

(iii) localized motion,

$$(u_I, v_I, h_I) \rightarrow 0, \quad x \rightarrow \pm\infty. \tag{2.9c}$$

In all cases the initial motion is assumed to be localized in the y-direction. First, formulae not depending on the type of the initial conditions will be derived, and then the special cases $(2.9a, b, c)$ are discussed.

Following RZB the solution to the system (2.1), (2.2), (2.3) is sought in the form of the following asymptotic expansions:

$$u = u_0(x, y, t, T_1, \ldots) + \varepsilon u_1(x, y, t, T_1, \ldots) + \cdots, \tag{2.10a}$$

$$v = v_0(x, y, t, T_1, \ldots) + \varepsilon v_1(x, y, t, T_1, \ldots) + \cdots, \tag{2.10b}$$

$$h = h_0(x, y, t, T_1, \ldots) + \varepsilon h_1(x, y, t, T_1, \ldots) + \cdots. \tag{2.10c}$$

Here $T_n = \varepsilon^n t$, $n = 1, 2, \ldots$ are the slow times needed to prevent the solution from a secular growth in time.

3. Lowest-order solution (linear adjustment)

3.1. *Splitting*

It is convenient to use the vorticity equation (2.5) instead of the mass conservation equation (2.1c) in the analysis to follow. Substitution of (2.10) into (2.1a, b), (2.2), (2.3), and (2.5) gives at the lowest order:

$$\frac{\partial u_0}{\partial t} - v_0 = -\frac{\partial h_0}{\partial x}, \quad \frac{\partial v_0}{\partial t} + u_0 = -\frac{\partial h_0}{\partial y}, \quad \frac{\partial (\zeta_0 - h_0)}{\partial t} = 0, \tag{3.1a–c}$$

$$v_0|_{y=0} = 0, \quad (u_0, v_0, h_0)_{t=0} = (u_I, v_I, h_I). \tag{3.1d, e}$$

By virtue of (3.1c) the lowest-order vorticity does not depend on the fast time t, i.e.

$$\zeta_0 - h_0 = \Pi_0(x, y, T_1, \ldots). \tag{3.2}$$

The solution to (3.1) is represented as a sum of fast and slow components,

$$(u_0, v_0, h_0) = (\tilde{u}_0, \tilde{v}_0, \tilde{h}_0)(x, y, t, T_1, \ldots) + (\bar{u}_0, \bar{v}_0, \bar{h}_0)(x, y, T_1, \ldots), \tag{3.3}$$

the fast component having zero average with respect to the fast time t, i.e.

$$\langle \tilde{u}_0 \rangle_t = \langle \tilde{v}_0 \rangle_t = \langle \tilde{h}_0 \rangle_t = 0. \tag{3.4}$$

Here the averaging is defined as follows:

$$\langle a \rangle_t = \lim \frac{1}{T_{av}} \int_0^{T_{av}} a \, dt \quad \text{as} \quad T_{av} \rightarrow \infty. \tag{3.5}$$

Applying the averaging (3.5) to (3.1a,b), (3.2) we obtain the following equations for the slow and fast components:

$$\bar{v}_0 = \frac{\partial \bar{h}_0}{\partial x}, \quad \bar{u}_0 = -\frac{\partial \bar{h}_0}{\partial y}, \quad \bar{\zeta}_0 - \bar{h}_0 = \nabla^2 \bar{h}_0 - \bar{h}_0 = \Pi_0, \tag{3.6a–c}$$

$$\bar{v}_0|_{y=0} = 0; \tag{3.6d}$$

$$\frac{\partial \tilde{u}_0}{\partial t} - \tilde{v}_0 = -\frac{\partial \tilde{h}_0}{\partial x}, \quad \frac{\partial \tilde{v}_0}{\partial t} + \tilde{u}_0 = -\frac{\partial \tilde{h}_0}{\partial y}, \quad \tilde{\zeta}_0 - \tilde{h}_0 = 0, \tag{3.7a–c}$$

$$\tilde{v}_0|_{y=0} = 0. \tag{3.7d}$$

3.2. Initial conditions for the fast and slow zonal velocities

Of course, the splitting (3.3) is incomplete unless the initial conditions for each component are determined. This can be easily done for the meridional velocity v_0. The initial field $\bar{v}_{0I}$ (in what follows the initial fields are designated by the subscript I) for the slow meridional velocity satisfies an equation readily derived from (3.6c) and the geostrophic equation (3.6a):

$$\nabla^2 \bar{v}_{0I} - \bar{v}_{0I} = \frac{\partial \Pi_I}{\partial x}, \quad \Pi_I = \zeta_I - h_I = \frac{\partial v_I}{\partial x} - \frac{\partial u_I}{\partial y} - h_I. \tag{3.8a, b}$$

Equation (3.8a) should be solved with the boundary condition

$$\bar{v}_{0I}|_{y=0} = 0. \tag{3.8c}$$

The system (3.8) allows us to find the initial field $\bar{v}_{0I}$. However, the initial field $\bar{h}_{0I}$ cannot be determined at this stage since (3.8c), (3.6a) imply only that $\bar{h}_{0I}$ is a constant at $y = 0$,

$$\bar{h}_{0I}|_{y=0} = \bar{h}_{0I}^{(B)} = \text{constant}, \tag{3.9}$$

and one has to determine this constant $\bar{h}_{0I}^{(B)}$, to make the problem for $\bar{h}_{0I}$ complete. Finding this constant is related to analysis of the Kelvin waves arising when the initial conditions (2.2) are not geostrophically balanced.

Knowing $\bar{v}_{0I}$ from (3.8) one can find the 'fast' initial field

$$\tilde{v}_{0I} = v_I - \bar{v}_{0I} = F(x, y). \tag{3.10a}$$

Another initial condition for $\tilde{v}_0$ follows from (3.6b), (3.7b), and (3.1e):

$$\left. \frac{\partial \tilde{v}_0}{\partial t} \right|_{t=0} = -\left(u_I + \frac{\partial h_I}{\partial y} \right) = G(x, y). \tag{3.10b}$$

The equation for $\tilde{v}_0$ is derived from (3.7a,b,c) (see e.g. RZB for details):

$$L_w \tilde{v}_0 = 0, \quad L_w = -\frac{\partial^2}{\partial t^2} + \nabla^2 - 1. \tag{3.11}$$

This equation together with the initial conditions (3.10), and the boundary condition (3.7d) allows us to determine the field $\tilde{v}_0$ given initial conditions (2.2).

3.3. *Fast fields* $\tilde{v}_0$, $\tilde{u}_0$, $\tilde{h}_0$

It is convenient to construct the solution for $\tilde{v}_0$ for the whole plane instead of the half-plane. To do that we determine the odd functions

$$(F^{odd}, G^{odd}) = \begin{cases} (F(x, y), G(x, y)), & y > 0 \\ -(F(x, -y), G(x, -y)), & y < 0 \end{cases} \tag{3.12}$$

and search for the solution to equation (3.11) which is bounded at infinity, and valid on the whole plane, with the initial conditions

$$\tilde{v}_{0I} = F^{odd}(x, y), \quad \left.\frac{\partial \tilde{v}_0}{\partial t}\right|_{t=0} = G^{odd}(x, y). \tag{3.13}$$

The resulting field $\tilde{v}_0$ is also odd and therefore satisfies the boundary condition (3.7d). The linear problem (3.11), (3.13) is conveniently solved using Fourier series (in the periodic case (2.9a)), and Fourier integrals. The resulting solutions in all cases are represented as a superposition of harmonic IG waves with constant amplitudes. In Appendix A the formulae for the solution to (3.11), (3.13) are presented for all cases (2.9a, b, c).

An important point is that the problem (3.11), (3.13) determines only the fast time dependence of $\tilde{v}_0$ and to introduce the slow time dependence we represent $\tilde{v}_0$ as a sum (cf. RZB):

$$\tilde{v}_0 = \tilde{v}_{0f}(x, y, t) + \tilde{v}_{00}(x, y, t, T_1, T_2, \ldots). \tag{3.14}$$

Here $\tilde{v}_{0f}$ is the solution to the problem (3.11), (3.13) and $\tilde{v}_{00}$ has a form analogous to $\tilde{v}_{0f}$ (i.e. it is also is a superposition of linear IG waves which is an odd function of y) but with the still unknown wave amplitudes depending on the slow times (see Appendix A and RZB for more details). Obviously, $\tilde{v}_{00}$ is a solution to the homogeneous wave equation (3.11) and the only condition to be satisfied by $\tilde{v}_{00}$ is that it is zero at $T_1 = T_2 = \ldots 0$. Due to this condition the correction $\tilde{v}_{00}$ can be neglected for times $t \ll \varepsilon^{-1}$. This additional term in the lowest-order wave solution and analogous terms in other fields are, in principle, necessary to avoid secular growth at higher order of the perturbation theory.

To find the fields $\tilde{u}_0$, $\tilde{h}_0$ we use the equations following from (3.7a, b, c):

$$\frac{\partial^2 \tilde{h}_0}{\partial y^2} - \tilde{h}_0 = -\left(\frac{\partial \tilde{v}_0}{\partial x} + \frac{\partial^2 \tilde{v}_0}{\partial t \partial y}\right), \quad \frac{\partial^2 \tilde{u}_0}{\partial y^2} - \tilde{u}_0 = \frac{\partial \tilde{v}_0}{\partial t} + \frac{\partial^2 \tilde{v}_0}{\partial x \partial y}. \tag{3.15a, b}$$

The solutions to (3.15a, b) are conveniently written in the form

$$\tilde{u}_0 = \tilde{u}_{01}(x, y, t, T_1, \ldots) + C_u^{(0)}(x, t, T_1, \ldots)\, e^{-y}, \tag{3.16a}$$

$$\tilde{h}_0 = \tilde{h}_{01}(x, y, t, T_1, \ldots) + C_h^{(0)}(x, t, T_1, \ldots)\, e^{-y}, \tag{3.16b}$$

where $C_u^{(0)}$ and $C_h^{(0)}$ are some arbitrary functions which have to be determined and $\tilde{u}_{01}$, $\tilde{h}_{01}$ are expressed in terms of $\tilde{v}_0$:

$$\tilde{u}_{01} = -\frac{1}{2}\left(\frac{\partial s^+}{\partial t} + \frac{\partial s^-}{\partial x}\right), \quad \tilde{h}_{01} = \frac{1}{2}\left(\frac{\partial s^+}{\partial x} + \frac{\partial s^-}{\partial t}\right), \tag{3.17a, b}$$

$$s^{\pm} = e^y \int_y^\infty \tilde{v}_0\, e^{-y}\, dy \pm e^{-y} \int_{-\infty}^y \tilde{v}_0\, e^y\, dy. \tag{3.17c}$$

The slow time dependence in $\tilde{u}_{01}$, $\tilde{h}_{01}$ is due to the term $\tilde{v}_{00}$ in (3.14).

3.4. *The Kelvin wave and the initial slow field $\bar{h}_{0I}$*

To determine the functions $C_u^{(0)}$, $C_h^{(0)}$ we write equations (3.7a, b, c) at $y = 0$:

$$\frac{\partial \tilde{u}_0}{\partial t} + \frac{\partial \tilde{h}_0}{\partial x} = 0, \quad \tilde{u}_0 = -\frac{\partial \tilde{h}_0}{\partial y}, \quad \tilde{h}_0 = -\frac{\partial \tilde{u}_0}{\partial y}, \quad y = 0, \qquad (3.18a\text{--}c)$$

Using the representation (3.17) and the fact that $\tilde{v}_0$ is an odd function of y, one can show that

$$\tilde{u}_0|_{y=0} = -\int_0^\infty \frac{\partial \tilde{v}_0}{\partial x} \, e^{-y} \, dy + C_u^{(0)}, \quad \left.\frac{\tilde{u}_0}{\partial y}\right|_{y=0} = -\int_0^\infty \frac{\partial \tilde{v}_0}{\partial t} \, e^{-y} \, dy - C_u^{(0)}; \qquad (3.19a, b)$$

$$\tilde{h}_0|_{y=0} = \int_0^\infty \frac{\partial \tilde{v}_0}{\partial t} \, e^{-y} \, dy + C_h^{(0)}, \quad \left.\frac{\partial \tilde{h}_0}{\partial y}\right|_{y=0} = \int_0^\infty \frac{\partial \tilde{v}_0}{\partial x} \, e^{-y} \, dy - C_h^{(0)}. \qquad (3.20a, b)$$

Substituting (3.19a), (3.20b) into (3.18b) we obtain that

$$C_u^{(0)} = C_h^{(0)}. \qquad (3.21)$$

In turn, the substitution of (3.19a), (3.20a) into (3.18a) and the taking into account of (3.21), gives the equation for $C_u^{(0)}$:

$$\frac{\partial C_u^{(0)}}{\partial t} + \frac{\partial C_u^{(0)}}{\partial x} = 0, \qquad (3.22)$$

whence

$$C_u^{(0)} = C_h^{(0)} = K_w^{(0)}(x - t, T_1, \ldots). \qquad (3.23)$$

Thus the last terms in (3.16a, b) describe the Kelvin wave propagating in such a way that the boundary is to the right of the propagation direction.

To determine the Kelvin wave profile we consider equation (3.20a) at the initial moment and use (3.10); as a result we have

$$\tilde{h}_{0I}|_{y=0} = -\int_0^\infty \left(u_I + \frac{\partial h_I}{\partial y} \right) e^{-y} \, dy + K_w^{(0)}(x). \qquad (3.24)$$

Bearing in mind that

$$\tilde{h}_{0I} + \bar{h}_{0I} = h_I \qquad (3.25)$$

we obtain the equation (see (3.9))

$$K_w^{(0)}(x) = \int_0^\infty \left(u_I + \frac{\partial h_I}{\partial y} \right) e^{-y} \, dy + h_I|_{y=0} - \bar{h}_{0I}|_{y=0}, \qquad (3.26)$$

relating the Kelvin wave profile to the constant boundary value $\bar{h}_{0I}^{(B)}$ (3.9) of the initial slow elevation $\bar{h}_{0I}$, which is also unknown. To calculate this constant and the Kelvin wave profile we use condition (3.4). As we show in Appendix B, the fields $\tilde{v}_0$, $\tilde{u}_{01}$, $\tilde{h}_{01}$ decay with increasing time t so that

$$\tilde{v}_0 = O\left(\frac{1}{t^{3/2}}\right), \quad \tilde{u}_{01} = O\left(\frac{1}{t^{3/2}}\right), \quad \tilde{h}_{01} = O\left(\frac{1}{t^{3/2}}\right), \quad t \to \infty, \quad x, y \text{ fixed} \quad (3.27)$$

and their averages (3.5) are definitely zero. Thus the restriction imposed by (3.4) on the Kelvin wave profile can be written as

$$\langle K_w^{(0)}(x - t) \rangle_t = \lim \frac{1}{T_{av}} \int_{x-T_{av}}^x K_w^{(0)}(z) \, dz = 0 \quad \text{as} \quad T_{av} \to \infty. \qquad (3.28)$$

Applying (3.28) to (3.26) one obtains the simple formulae for the boundary value of the initial slow elevation

$$\bar{h}_{0I}^{(B)} = \int_0^\infty \langle u_I + h_I \rangle | e^{-y}\, dy, \tag{3.29}$$

and the Kelvin wave profile

$$K_w^{(0)}(x) = \int_0^\infty [u_I + h_I - \langle u_I + h_I \rangle] e^{-y}\, dy \tag{3.30}$$

where the averaging

$$\langle a(x) | = \lim \frac{1}{T_{av}} \int_{x=T_{av}}^x a(x)\, dx \quad \text{as} \quad T_{av} \to \infty \tag{3.31}$$

is introduced. Of course, if $a(x)$ is a smooth bounded function (as our initial fields are assumed to be) then $\langle a \rangle$ does not depend on x and depends only on the behaviour of $a(x)$ as $x \to -\infty$. Physically the condition (3.29) means that the fast Kelvin wave brings 'information' from $x = -\infty$ to $x = +\infty$ propagating always in such a way that the boundary is to the right of the propagation direction. Correspondingly, the initial boundary condition for the slow component is determined only by the initial fields at $x = -\infty$.

Formula (3.30) gives us a simple way to calculate the lowest-order Kelvin wave profile for arbitrary initial conditions (2.2). In the case of periodic boundary conditions (2.9a) the condition (3.30) means that the periodic Kelvin wave profile should not contain a part independent of x (purely zonal flow), i.e.

$$K_w^{(0)}(x) = \int_0^\infty [u_I + h_I - u_{I0} - h_{I0}]\, e^{-y}\, dy \tag{3.32a}$$

and for $\bar{h}_{0I}^{(B)}$ we have

$$\bar{h}_{0I}^{(B)} = \int_0^\infty (u_{I0} + h_{I0})\, e^{-y}\, dy. \tag{3.32b}$$

In the case of an initial 'zonal step' (2.9b) the Kelvin wave profile and $\bar{h}_{0I}^{(B)}$ are given by

$$K_w^{(0)}(x) = \int_0^\infty [u_I + h_I - u_I^{(-)} - h_I^{(-)}]\, e^{-y}\, dy, \tag{3.33a}$$

$$\bar{h}_{0I}^{(B)} = \int_0^\infty [u_I^{(-)} + h_I^{(-)}]\, e^{-y}\, dy. \tag{3.33b}$$

Finally, in the localized case (2.9c) we have

$$K_w^{(0)}(x) = \int_0^\infty (u_I + h_I)\, e^{-y} dy, \tag{3.34a}$$

$$\bar{h}_{0I}^{(B)} = 0. \tag{3.34b}$$

The Kelvin wave profiles for each of these cases are shown schematically in figure 2.

Note that in the case of geostrophically balanced initial conditions when $h_I = \bar{h}_{0I}$ and $u_I = -\partial h_I / \partial y$ the Kelvin wave disappears, as follows from (3.26). In the absence of nonlinearity the initial Kelvin wave profile (3.30) propagates steadily; the nonlinearity forces the profile to change slowly in time (see below, §5).

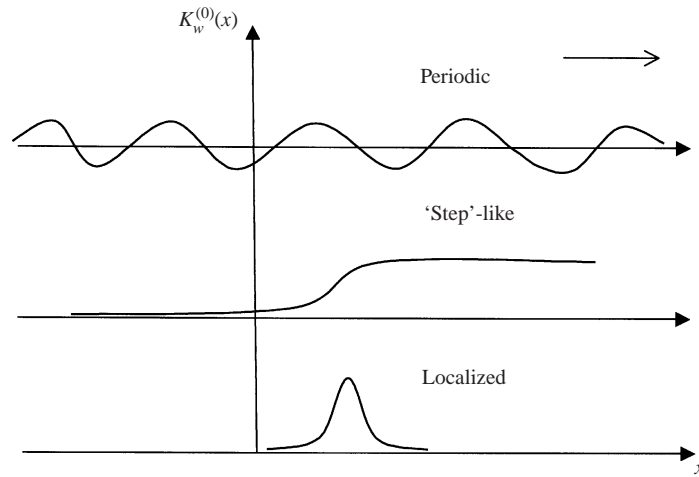Knowing the boundary condition (3.29) (or (3.32b), (3.33b), (3.34b)) one can find

FIGURE 2. Schematic representation of the lowest-order initial Kelvin wave profile $K_w^{(0)}(x)$ for various initial conditions.

$\bar{h}_{0I}$ using the equation

$$\nabla^2 \bar{h}_{0I} - \bar{h}_{0I} = \Pi_I = \frac{\partial v_I}{\partial x} - \frac{\partial u_I}{\partial y} - h_I, \qquad (3.35)$$

which readily follows from the initial conditions (3.1e) and equations (3.3), (3.6c), (3.7c). Clearly, the solution $\bar{h}_{0I}, \tilde{u}_0, \tilde{v}_0, \tilde{h}_0$ (without the slow time dependence) describes the *linear* adjustment of an arbitrary initial field (2.2) on the half-plane.

Thus, at the lowest order we obtain a fast–slow motion splitting defined in a unique way for arbitrary initial conditions. Note that the procedure imposes no limitations on the relative initial values of fast and slow components. The fast part of the motion is completely resolved while the slow one remains undetermined. The slow evolution is determined from the condition of the absence of secular growth of the first-order solution.

4. Dynamics of the lowest-order slow motion
4.1. *Problem description for the lowest-order slow motion*

To describe the time development of the lowest-order slow component and slow evolution of the fast one we consider the first-order solution. Substitution of (2.10) into (2.1a, b), (2.2), (2.3), (2.5) gives

$$\frac{\partial u_1}{\partial t} - v_1 = -\frac{\partial h_1}{\partial x} + R_u^{(1)}, \quad \frac{\partial v_1}{\partial t} + u_1 = -\frac{\partial h_1}{\partial y} + R_v^{(1)}, \quad \frac{\partial(\zeta_1 - h_1)}{\partial t} = R_\zeta^{(1)}, \quad (4.1a\text{--}c)$$

$$R_u^{(1)} = -\left(\frac{\partial u_0}{\partial T_1} + u_0 \frac{\partial u_0}{\partial x} + v_0 \frac{\partial u_0}{\partial y}\right), \quad R_v^{(1)} = -\left(\frac{\partial v_0}{\partial T_1} + u_0 \frac{\partial v_0}{\partial x} + v_0 \frac{\partial v_0}{\partial y}\right), \quad (4.1d, e)$$

$$R_\zeta^{(1)} = -\frac{\partial(\zeta_0 - h_0)}{\partial T_1} - \frac{\partial[u_0(\zeta_0 - h_0)]}{\partial x} - \frac{\partial[v_0(\zeta_0 - h_0)]}{\partial y}, \qquad (4.1f)$$

$$v_1|_{y=0} = 0, \quad (u_1, v_1, h_1)_{t=0} = 0. \qquad (4.1g, h)$$

Analysis of system (4.1) is carried out along the same lines as in the lowest-order case. We represent the first-order field as a sum of fast and slow components,

$$(u_1, v_1, h_1) = (\tilde{u}_1, \tilde{v}_1, \tilde{h}_1)(x, y, t, T_1, \ldots) + (\bar{u}_1, \bar{v}_1, \bar{h}_1)(x, y, T_1, \ldots), \qquad (4.2)$$

the fast component having zero average (3.5), and apply (3.5) to equations (4.1).

The averaging of the first-order vorticity equation (4.1c) gives the QG potential vorticity equation

$$\frac{\partial(\nabla^2 \bar{h}_0 - \bar{h}_0)}{\partial T_1} + J(\bar{h}_0, \nabla^2 \bar{h}_0) = 0 \qquad (4.3)$$

describing evolution of the slow motion. The first-order slow equations are written in the form

$$\bar{v}_1 = \frac{\partial \bar{h}_1}{\partial x} - \overline{R}_u^{(1)}, \quad \bar{u}_1 = -\frac{\partial \bar{h}_1}{\partial y} + \overline{R}_v^{(1)}, \quad \zeta - \bar{h}_1 = \Pi_1(x, y, T_1, \ldots), \qquad (4.4a\text{–}c)$$

$$\bar{v}_1|_{y=0} = 0; \qquad (4.4d)$$

$$\overline{R}_u^{(1)} = -\left(\frac{\partial \bar{u}_0}{\partial T_1} + \bar{u}_0 \frac{\partial \bar{u}_0}{\partial x} + \bar{v}_0 \frac{\partial \bar{u}_0}{\partial y} \right), \quad \overline{R}_v^{(1)} = -\left(\frac{\partial \bar{v}_0}{\partial T_1} + \bar{u}_0 \frac{\partial \bar{v}_0}{\partial x} + \bar{v}_0 \frac{\partial \bar{v}_0}{\partial y} \right). \qquad (4.4e,f)$$

Here Π_1 is some function of the slow variables which is determined at the next order. Note that the averages of the nonlinear terms containing the lowest-order fast fields $\tilde{u}_0, \tilde{v}_0$ are zero, which readily follows from the large-time decay of the lowest-order fast fields (see (3.27)) and the property (3.28) of Kelvin waves.

The initial field $\bar{h}_{0I}$ is determined by equations (3.35), (3.29), and the boundary value of $\bar{h}_0$ depends only on the slow time by virtue of (3.6a, d),

$$\bar{h}_0|_{y=0} = \bar{h}_{0B}(T_1, \ldots). \qquad (4.5)$$

To find the function $\bar{h}_{0B}(T_1, \ldots)$ in (4.5) we use the boundary condition for the first-order slow variable $\bar{h}_1$ that readily follows from equations (4.4a), (4.4d), and (4.4e):

$$\left. \frac{\partial \bar{h}_1}{\partial x} \right|_{y=0} = -\left(\frac{\partial \bar{u}_0}{\partial T_1} + \bar{u}_0 \frac{\partial \bar{u}_0}{\partial x} \right)_{y=0}. \qquad (4.6)$$

The function $\bar{h}_1$ should be bounded as $x \to \pm\infty$; therefore the condition (4.6) imposes the following additional restriction on the lowest-order fields:

$$\left\langle \left. \frac{\partial \bar{u}_0}{\partial T_1} \right|_{y=0} \right\rangle_x = -\left\langle \left. \frac{\partial^2 \bar{h}_0}{\partial y \, \partial T_1} \right|_{y=0} \right\rangle_x = 0, \qquad (4.7)$$

where the averaging $\langle \ \rangle_x$ is determined from

$$\langle a \rangle_x = \lim \frac{1}{2L} \int_{-L}^{L} a \, \mathrm{d}x, \quad L \to \infty. \qquad (4.8)$$

The condition (4.7) of constant mean circulation along the boundary is well-known (see e.g. Kamenkovich & Reznik 1978; Pedlosky 1987) but previously it was obtained for geostrophically balanced initial conditions (2.2), i.e. no fast component was present in the system.

4.2. *Solvability of the problem for* $\bar{h}_0$

The problem (4.3), (4.5), and (4.7) together with initial field $\bar{h}_{0I}$ known from (3.29), (3.35) is complete and allows us to determine the lowest-order slow geostrophic component of the motion.

To demonstrate the solvability of this problem we rewrite (4.3) in the form

$$\nabla^2 D - D = R_D, \quad D = \frac{\partial \bar{h}_0}{\partial T_1}, \quad R_D = -J(\bar{h}_0, \nabla^2 \bar{h}_0), \tag{4.9}$$

and assume $\bar{h}_0$ to be known at some moment $T_1 = T_{10}$. We now represent D as a sum

$$D = D_0 + D_1 \tag{4.10}$$

where D_0, D_1 satisfy the equations

$$\nabla^2 D_0 - D_0 = R_D, \quad D_0|_{y=0} = 0; \tag{4.11}$$

$$\nabla^2 D_1 - D_1 = 0, \quad D_1|_{y=0} = \frac{\partial \bar{h}_{0B}}{\partial T_1}. \tag{4.12}$$

Here R_D and, therefore, D_0 are known by the above assumption, and $\partial \bar{h}_{0B}/\partial T_1$ can be found. Since $\partial \bar{h}_{0B}/\partial T_1$ does not depend on x we have

$$D_1 = \frac{\partial \bar{h}_{0B}}{\partial T_1} e^{-y}, \tag{4.13}$$

and substitution of (4.10) into the condition (4.7) gives

$$\frac{\partial \bar{h}_{0B}}{\partial T_1} = \left\langle \frac{\partial D_0}{\partial y}\bigg|_{y=0} \right\rangle_x \tag{4.14}$$

whence

$$\frac{\partial \bar{h}_0}{\partial T_1} = D_0 + e^{-y} \left\langle \frac{\partial D_0}{\partial y}\bigg|_{y=0} \right\rangle_x. \tag{4.15}$$

Given $\partial \bar{h}_0/\partial T_1$ one can calculate the field $\bar{h}_0$ at $T_1 = T_{10} + dT_1$, and so on.

For the zonal 'step' case (2.9b) the boundary condition (4.5) (i.e. the function $\bar{h}_{0B}$) does not depend on time and the slow motion does not change as $x \to \mp\infty$, being approximately zonal there. To show this, we note that if at some moment $T_1 = T_{10}$

$$\bar{h}_0 \to \bar{h}_0^\pm(y), \quad x \to \pm\infty, \tag{4.16}$$

then the derivative $\partial \bar{h}_0/\partial x$ and Jacobian R_D are localized in x. In this case the solution D_0 to the problem (4.11) is also localized in x, i.e. $\langle \partial D_0/\partial y|_{y=0}\rangle_x = 0$. This means that $\partial \bar{h}_{0B}/\partial T_1 = 0$ (see (4.14)) and the derivative $\partial \bar{h}_0/\partial T_1 = D_0$ is also localized in x at the moment $T_1 = T_{10}$. Thus $\bar{h}_0$ does not change as $x \to \mp\infty$ and the property (4.16) remains valid at the moment $T_1 = T_{10} + dT_1$ and so on. By virtue of (2.9b) we have for the initial vorticity Π_I (right-hand side of equation (3.35)):

$$\Pi_I \to -\frac{\partial u_I^{(\pm)}(y)}{\partial y} - h_I^{(\pm)}(y) \quad \text{as} \quad x \to \pm\infty; \tag{4.17}$$

therefore the initial field $\bar{h}_{0I}$ also obeys (4.16). Thus for the zonal-step initial condition the slow component remains unchanged as $x \to \mp\infty$, and is described by the QG

equation (4.3) with the time-independent boundary condition

$$\bar{h}_{0B} = \int_0^\infty [u_I^{(-)} + h_I^{(-)}] \, e^{-y} \, dy = \text{constant.} \tag{4.18}$$

This consideration remains valid for the localized initial conditions (2.9c) which are special, but a practically important case of the step initial conditions. By virtue of (3.34b), (4.18) we have $\bar{h}_{0B} = 0$ in the localized case, and, therefore (see (4.5)), the evolution of the localized initial field $\bar{h}_{0I}$ is governed by (4.3) under the condition

$$\bar{h}_0|_{y=0} = 0, \tag{4.19}$$

i.e. $\bar{h}_0$ remains localized for all time. This means that that the right-hand side of (4.6) is also localized and the first-order correction $\bar{h}_1$ is always bounded. Correspondingly, the condition (4.7) for the localized case is superfluous since it is satisfied identically, and imposes no additional restrictions on the motion.

4.3. *Mass, energy, and enstrophy conservation*

Mass and energy conservation for the periodic and step cases are obtained in the usual way and have the forms

$$\frac{\partial}{\partial T_1} \int_0^\infty \langle \bar{h}_0 \rangle_x \, dy = - \left\langle \frac{\partial^2 \bar{h}_0}{\partial y \, \partial T_1} \bigg|_{y=0} \right\rangle_x = 0 \tag{4.20a}$$

and

$$\frac{\partial}{\partial T_1} \int_0^\infty \langle \tfrac{1}{2} [(\nabla \bar{h}_0)^2 + \bar{h}_0^2] \rangle_x \, dy = -\bar{h}_{0B} \left\langle \frac{\partial^2 \bar{h}_0}{\partial y \, \partial T_1} \bigg|_{y=0} \right\rangle_x = 0, \tag{4.20b}$$

respectively. Thus the condition (4.7) provides the energy and mass conservation for the lowest-order slow motion. Multiplying (4.3) by $\nabla^2 \bar{h}_0$, averaging in x and integrating in y from 0 to ∞ we obtain the enstrophy conservation law:

$$\frac{\partial}{\partial T_1} \left\{ \int_0^\infty \langle \tfrac{1}{2} [(\nabla \bar{h}_0)^2 + (\nabla^2 \bar{h}_0)^2] \rangle_x \, dy + \Gamma \, \bar{h}_{0B} \right\} = 0, \tag{4.20c}$$

where $\Gamma = \langle \partial \bar{h}_0 / \partial y |_{y=0} \rangle_x = \text{constant}$ by virtue of (4.7). Note that the law (4.20c) includes the extra boundary term $\Gamma \bar{h}_{0B}$ absent in an unbounded fluid.

For the localized case, the energy and enstrophy conservation laws have a standard form, and the mass conservation is written as

$$\frac{\partial}{\partial T_1} \int_{-\infty}^\infty dx \int_0^\infty \bar{h}_0 \, dy = - \int \frac{\partial^2 \bar{h}_0}{\partial y \, \partial T_1} \bigg|_{y=0} \, dx. \tag{4.21}$$

It readily follows from (4.21) that the mass is conserved only under the condition of zero along-boundary circulation,

$$\int_{-\infty}^\infty \frac{\partial^2 \bar{h}_0}{\partial y \, \partial T_1} \bigg|_{y=0} \, dx = 0. \tag{4.22}$$

Note that the equation (4.22) is also a condition for locality of the first-order slow correction $\bar{h}_1$, as is readily seen from (4.6). But the problem (4.3), (4.19) is well-posed and the condition (4.22) turns out to be superfluous, i.e. it can contradict (4.19). To

demonstrate this we use the formula

$$\int_{-\infty}^{\infty} \frac{\partial^2 \bar{h}_0}{\partial y \, \partial T_1}\bigg|_{y=0} \mathrm{d}x = \int_{-\infty}^{\infty} \mathrm{d}x \int_0^{\infty} \frac{\partial \bar{h}_0}{\partial x} \frac{\partial \bar{h}_0}{\partial y} \, \mathrm{e}^{-y} \mathrm{d}y, \qquad (4.23)$$

which follows from (4.3) and (4.19). To derive (4.23) we integrate (4.3) over all x; as a result we have

$$\frac{\partial^2 H}{\partial y^2} - H = -\frac{\partial}{\partial y} \int_{-\infty}^{\infty} \frac{\partial \bar{h}_0}{\partial x} \frac{\partial^2 \bar{h}_0}{\partial y^2} \, \mathrm{d}x, \quad H = \int_{-\infty}^{\infty} \frac{\partial \bar{h}_0}{\partial T_1} \, \mathrm{d}x. \qquad (4.24a,b)$$

Multiplying (4.24a) by e^{-y} and integrating the resulting equation over all y with the boundary condition (4.19) we arrive at (4.23). Generally, the integral on the right-hand side of (4.23) does not vanish for an arbitrary localized $\bar{h}_0$ satisfying (4.19). This means that the condition (4.22) can be violated, at least at the initial moment $T_1 = 0$. Therefore equations (4.3), (4.19), generally, do not validate (4.22), which means (somewhat unexpectedly) that in the case of localized initial conditions (2.9c): (i) the mass of the slow localized motion on a bounded half-plane may not be conserved; (ii) the first-order slow correction field may not be localized.

5. First-order solution
5.1. *Problem description for the fast first-order component*
For the fast first-order components we have (cf. RZB)

$$\frac{\partial \tilde{u}_1}{\partial t} - \tilde{v}_1 = -\frac{\partial \tilde{h}_1}{\partial x} + \tilde{R}_u^{(1)}, \quad \frac{\partial \tilde{v}_1}{\partial t} + \tilde{u}_1 = -\frac{\partial \tilde{h}_1}{\partial y} + \tilde{R}_v^{(1)}, \quad \tilde{\zeta}_1 - \tilde{h}_1 = \tilde{R}_\zeta^{(1)}, \qquad (5.1a\text{--}c)$$

$$\tilde{v}_1|_{y=0} = 0. \qquad (5.1d)$$

Here

$$\tilde{R}_u^{(1)} = -\left(\frac{\partial \tilde{u}_0}{\partial T_1} + \tilde{u}_0 \frac{\partial \tilde{u}_0}{\partial x} + \tilde{v}_0 \frac{\partial \tilde{u}_0}{\partial y} + \bar{u}_0 \frac{\partial \tilde{u}_0}{\partial x} + \bar{v}_0 \frac{\partial \tilde{u}_0}{\partial y} + \tilde{u}_0 \frac{\partial \bar{u}_0}{\partial x} + \tilde{v}_0 \frac{\partial \bar{u}_0}{\partial y} \right), \qquad (5.1e)$$

$$\tilde{R}_v^{(1)} = -\left(\frac{\partial \tilde{v}_0}{\partial T_1} + \tilde{u}_0 \frac{\partial \tilde{v}_0}{\partial x} + \tilde{v}_0 \frac{\partial \tilde{v}_0}{\partial y} + \bar{u}_0 \frac{\partial \tilde{v}_0}{\partial x} + \bar{v}_0 \frac{\partial \tilde{v}_0}{\partial y} + \tilde{u}_0 \frac{\partial \bar{v}_0}{\partial x} + \tilde{v}_0 \frac{\partial \bar{v}_0}{\partial y} \right), \qquad (5.1f)$$

$$\tilde{R}_\zeta^{(1)} = -\frac{\partial [\tilde{U}_0 (\zeta_0 - h_0)]}{\partial x} - \frac{\partial [\tilde{V}_0 (\zeta_0 - h_0)]}{\partial y}, \qquad (5.1g)$$

$$\tilde{U}_0 = \tilde{U}_{0i} - \langle \tilde{U}_{0i} \rangle_t, \quad \tilde{V}_0 = \tilde{V}_{0i} - \langle \tilde{V}_{0i} \rangle_t, \quad (\tilde{U}_{0i}, \tilde{V}_{0i}) = \int_0^t (\tilde{u}_0, \tilde{v}_0) \, \mathrm{d}t. \qquad (5.1h)$$

To derive equation (5.1c) we integrate (4.1c) over t from 0 to t taking into account (4.1h) and, using the averaging procedure (3.5), split the resulting equation into the slow and fast parts. The function $\tilde{V}_0$ is determined from the system readily obtained from equation (3.11) and conditions (3.7d), (3.10a,b):

$$L_w \tilde{V}_0 = 0, \quad \tilde{V}_0|_{y=0} = 0, \quad \tilde{V}_0|_{t=0} = -\langle \tilde{V}_{0i} \rangle_t, \quad \frac{\partial \tilde{V}_0}{\partial t}\bigg|_{t=0} = F(x,y), \qquad (5.2a\text{--}d)$$

where the function $\langle \tilde{V}_{0i} \rangle_t$ obeys the equations

$$\nabla^2 \langle \tilde{V}_{0i} \rangle_t - \langle \tilde{V}_{0i} \rangle_t = -G(x,y), \quad \langle \tilde{V}_{0i} \rangle_t|_{y=0} = 0. \qquad (5.3)$$

Given $\tilde{V}_0$ the function $\tilde{U}_0$ is determined by the formulae

$$\tilde{U}_0 = \tilde{U}_{01} + \tilde{U}_{0k}, \quad \frac{\partial U_{0k}}{\partial t} = K_w^{(0)}(x - t)\,\mathrm{e}^{-y}, \tag{5.4a}$$

where

$$\tilde{U}_{01} = -\frac{s^+}{2} - \frac{1}{2}\frac{\partial}{\partial x}\left(\mathrm{e}^y \int_y^\infty \tilde{V}_0\,\mathrm{e}^{-y}\,\mathrm{d}y - \mathrm{e}^{-y}\int_{-\infty}^y \tilde{V}_0\,\mathrm{e}^y\,\mathrm{d}y\right). \tag{5.4b}$$

The function $\tilde{U}_{0k}$ is given by the equation

$$\tilde{U}_{0k} = -\mathrm{e}^{-y}\int_{-\infty}^{x-t} K_w^{(0)}(z, T_1, \ldots)\,\mathrm{d}z, \tag{5.4c}$$

in the 'step' and localized cases, and

$$\tilde{U}_{0k} = \mathrm{i}\mathrm{e}^{-y}\sum_m \frac{k_m(T_1, \ldots)}{m}\,\mathrm{e}^{\mathrm{i}m(x-t)}, \quad k_0 = 0, \tag{5.4d}$$

in the periodic case; here k_m are the coefficient in the Fourier-series expansion in x of the Kelvin wave $K_w^{(0)}(x - t, T_1, \ldots)\,\mathrm{e}^{-y}$.

5.2. *Initial conditions for the fast and slow first-order zonal velocities*

Like the lowest-order approximation (§ 3) the analysis starts with calculating the initial slow first-order meridional velocity $\bar{v}_{1I}$. First, we derive from (4.4a, b, c) the equation for $\partial \bar{h}_{1I}/\partial x$ analogous to (3.8),

$$\nabla^2 \frac{\partial \bar{h}_{1I}}{\partial x} - \frac{\partial \bar{h}_{1I}}{\partial x} = \frac{\partial}{\partial x}[\Pi_{1I} + \mathrm{div}\overline{\boldsymbol{R}}_{vI}^{(1)}], \tag{5.5a}$$

where

$$\overline{\boldsymbol{R}}_v^{(1)} = (\overline{R}^{(1)u}, \overline{R}_v^{(1)}), \quad \mathrm{div}\overline{\boldsymbol{R}}_v^{(1)} = 2\left[\frac{\partial^2 \bar{h}_0}{\partial x^2}\frac{\partial^2 \bar{h}_0}{\partial y^2} - \left(\frac{\partial^2 \bar{h}_0}{\partial x\,\partial y}\right)^2\right]. \tag{5.5b, c}$$

The boundary condition for (5.5a) follows from (4.6):

$$\left.\frac{\partial \bar{h}_{1I}}{\partial x}\right|_{y=0} = -\left(\left.\frac{\partial \bar{u}_0}{\partial T_1}\right|_{t=0} + \bar{u}_{0I}\frac{\partial \bar{u}_{0I}}{\partial x}\right)_{y=0}. \tag{5.6}$$

To complete the problem (5.5a), (5.6) Π_{1I} and $\partial \bar{u}_0/\partial T_1|_{t=0}$ should be found. By virtue of (4.4c), (5.1c) $\zeta_1 - h_1 = \Pi_1 + \tilde{R}_\zeta^{(1)}$, and, therefore, taking into account initial conditions (4.1h), we have

$$\Pi_{1I} = -\tilde{R}_{\zeta I}^{(1)} = \frac{\partial(\tilde{U}_{0I}\Pi_I)}{\partial x} + \frac{\partial(\tilde{V}_{0I}\Pi_I)}{\partial y}, \tag{5.7}$$

where $\tilde{U}_{0I}$, $\tilde{V}_{0I}$ can be found using (5.2) to (5.4). The derivative $\partial \bar{u}_0/\partial T_1|_{t=0}$ is obtained from (4.15),

$$\left.\frac{\partial \bar{u}_0}{\partial T_1}\right|_{t=0} = -\left.\frac{\partial^2 \bar{h}_0}{\partial y\,\partial T_1}\right|_{t=0} = -\frac{\partial D_{0I}}{\partial y} + \mathrm{e}^{-y}\left\langle\left.\frac{\partial D_{0I}}{\partial y}\right|_{y=0}\right\rangle_x. \tag{5.8}$$

Thus the problem for $\partial \bar{h}_{1I}/\partial x$ is well-defined and given $\partial \bar{h}_{1I}/\partial x$ the initial field $\bar{v}_{1I}$ can be determined from (4.4a, e):

$$\bar{v}_{1I} = \frac{\partial \bar{h}_{1I}}{\partial x} - \overline{R}_{uI}^{(1)}. \tag{5.9}$$

The function $\bar{v}_{1I}$ is periodic for the periodic case (2.9*a*), and localized for the cases (2.9*b*, *c*). Given $\bar{v}_{1I}$ one can determine one initial condition $\tilde{v}_{1I}$ from (4.1*h*):

$$\tilde{v}_{1I} = -\tilde{v}_{1I}. \tag{5.10}$$

Another initial condition for $\tilde{v}_1$ follows from (4.1*b*, *h*):

$$\left.\frac{\partial \tilde{v}_1}{\partial t}\right|_{t=0} = R_{vI}^{(1)} = -\left(\left.\frac{\partial v_0}{\partial T_1}\right|_{t=0} + u_I \frac{\partial v_I}{\partial x} + v_I \frac{\partial v_I}{\partial y}\right). \tag{5.11}$$

As we will see below, $\tilde{v}_0$ does not depend on the slow time T_1, therefore from (4.15),

$$\left.\frac{\partial v_0}{\partial T_1}\right|_{t=0} = \left.\frac{\partial^2 \bar{h}_0}{\partial x \, \partial T_1}\right|_{t=0} = \frac{\partial D_{0I}}{\partial x}. \tag{5.12}$$

5.3. *Analysis of resonances in the equation for $\tilde{v}_1$*

The equation for $\tilde{v}_1$ follows from (5.1*a*, *b*, *c*) (see RZB for more details):

$$-\frac{\partial^2 \tilde{v}_1}{\partial t^2} + \nabla^2 \tilde{v}_1 - \tilde{v}_1 = \tilde{F}_v^{(1)}, \tag{5.13a}$$

$$\tilde{F}_v^{(1)} = -\left(\frac{\partial^2 \tilde{R}_u^{(1)}}{\partial y^2} - \tilde{R}_u^{(1)} + \frac{\partial \tilde{R}_v^{(1)}}{\partial t} - \frac{\partial^2 \tilde{R}_v^{(1)}}{\partial x \, \partial y} + \frac{\partial^2 \tilde{R}_\zeta^{(1)}}{\partial t \, \partial y} - \frac{\partial \tilde{R}_\zeta^{(1)}}{\partial x}\right). \tag{5.13b}$$

This equation, together with the initial conditions (5.10), (5.11) and the boundary condition (5.1*d*), determine the field $\tilde{v}_1$.

Of course, the first question that arises is whether the right-hand side of (5.13*a*) contains secular terms causing either a rapidly growing response, or a response with a non-zero average (3.5) (in this case our first-order splitting (4.2) fails). To examine this issue the source term $\tilde{F}_v^{(1)}$ is represented as a sum (using (3.15*b*), cf. RZB):

$$\tilde{F}_v^{(1)} = 2\frac{\partial^2 \tilde{v}_0}{\partial T_1 \, \partial t} + \Phi_0^{(s)}\Phi_0^{(ig)} + \Phi_1^{(ig)}\Phi_2^{(ig)} + \Phi_0^{(k)}\Phi_3^{(ig)} + \Phi_1^{(k)}\Phi_1^{(s)} + \Phi_2^{(k)}\Phi_3^{(k)}, \tag{5.14}$$

where the superscripts *s*, *ig*, and *k* denote the slow component, the component consisting of the IG waves, the Kelvin wave component, respectively. Each of the functions $\Phi_m^{(ig)}, \Phi_m^{(k)}$, $m = 0, 1, 2, 3$, is a solution to the homogeneous wave equation (3.11); the slow functions $\Phi_n^{(s)}$, $n = 0, 1$, do not depend on the fast time *t*. The functions $\Phi_n^{(s)}, \Phi_m^{(ig)}, \Phi_m^{(k)}$ are localized in the *y*-direction, and either periodic or bounded or localized in the *x*-direction depending on the initial conditions (2.9). Analysis of all possible interactions is rather tedious; some estimates for the localized case are given in Appendix B. Here only the results will be reported.

Triad interactions between IG waves are prohibited, as can be readily shown using their dispersion relation (*A*10*a*), and, therefore the response $\tilde{v}_{ww}$ generated by the term $\Phi_1^{(ig)}\Phi_2^{(ig)}$ decays rapidly, $\tilde{v}_{ww}|_{t=\infty} = O(t^{-3})$ at a fixed point *x*, *y*. The interactions between Kelvin waves are contained in the first two terms in the brackets in the right-hand side of (5.13*b*) and the corresponding source term $\Phi_2^{(k)}\Phi_3^{(k)}$ has the form

$$\Phi_2^{(k)}\Phi_3^{(k)} = 3K_w^{(0)}(x-t, T_1, \ldots)\frac{\partial K_w^{(0)}}{\partial x}(x-t, T_1, \ldots)\,\mathrm{e}^{-2y}. \tag{5.15}$$

The corresponding response

$$\tilde{v}_{kk} = K_w^{(0)}(x-t, T_1, \ldots)\frac{\partial K_w^{(0)}}{\partial x}(x-t, T_1, \ldots)\,\mathrm{e}^{-2y} \tag{5.16}$$

does not decay in time but it is fast since

$$\langle \tilde{v}_{kk} \rangle_t = 0, \tag{5.17}$$

and makes no contribution to the slow component.

The responses $\tilde{v}_{sw}$ and $\tilde{v}_{kw}$ to the interactions between the slow component and IG waves (term $\Phi_0^{(s)} \Phi_0^{(ig)}$), and between Kelvin and IG waves (term $\Phi_0^{(k)} \Phi_3^{(ig)}$), respectively, decay in time but the decay rate depends on the initial conditions. For the periodic initial conditions both $\tilde{v}_{sw}$ and $\tilde{v}_{kw}$ are $O(t^{-1/2})$, $t \to \infty$, while for the step and localized cases $\tilde{v}_{sw} = O(t^{-1})$, $\tilde{v}_{kw} = O(t^{-3/2})$. Finally, the response $\tilde{v}_{sk}$ to the interaction between Kelvin waves and the slow motion (term $\Phi_1^{(k)} \Phi_1^{(s)}$) does not decay for the periodic case, but $\langle \tilde{v}_{sk} \rangle_t = 0$, and $\tilde{v}_{sk} = O(t^{-1})$ for the step and localized cases. We note that although the Fourier representations of the nonlinear terms $\Phi_0^{(s)} \Phi_0^{(ig)}$, $\Phi_0^{(k)} \Phi_3^{(ig)}$, and $\Phi_1^{(k)} \Phi_1^{(s)}$, contain resonant harmonics their measure is zero in the continuous spectrum and these harmonics give no dangerous contributions (cf. RZB).

The general conclusion of this analysis is that the response to nonlinear interactions in (5.14) does not grow in time, and it is fast in the sense that it has a zero time-average (3.5). We emphasize that taking into account the boundary and initial conditions (5.1*d*), (5.10), (5.11) does not change this conclusion.

The lack of the resonant nonlinear terms on the right-hand side of (5.13*a*) means that $\tilde{v}_0$, and therefore, the function $\tilde{v}_{00}$ in (3.14) do not depend on the slow time T_1 since in the opposite case the first term on the right hand side of (5.14) gives a secular growth. Clearly, the functions $\tilde{u}_{01}$, $\tilde{h}_{01}$ in (3.16) also do not depend on T_1, since they are linearly depend on $\tilde{v}_0$ (see §3). At the same time, as we will see below, the lowest-order Kelvin wave in (3.16) does depend on slow times to prevent secular growth of the first-order Kelvin wave.

5.4. *The fields $\tilde{u}_1$, $\tilde{h}_1$, and the first-order Kelvin wave*

The fields $\tilde{u}_1$, $\tilde{h}_1$ are found analogously to $\tilde{u}_0$, $\tilde{h}_0$ in §3. The equations analogous to (3.15) simply follow from (5.1*a,b,c*):

$$\frac{\partial^2 \tilde{h}_1}{\partial y^2} - \tilde{h}_1 = \tilde{F}_h^{(1)} = -\left(\frac{\partial \tilde{v}_1}{\partial x} + \frac{\partial^2 \tilde{v}_1}{\partial t \, \partial y} \right) + \tilde{R}_\zeta^{(1)} + \frac{\partial \tilde{R}_v^{(1)}}{\partial y}, \tag{5.18a}$$

$$\frac{\partial^2 \tilde{u}_1}{\partial y^2} - \tilde{u}_1 = \tilde{F}_u^{(1)} = -\frac{\partial \tilde{v}_1}{\partial t} + \frac{\partial^2 \tilde{v}_1}{\partial x \, \partial y} - \tilde{R}_v^{(1)} - \frac{\partial \tilde{R}_\zeta^{(1)}}{\partial y}, \tag{5.18b}$$

Solutions to (5.18*a,b*) are conveniently written in the form

$$\tilde{u}_1 = \tilde{u}_{11} + C_u^{(1)}(x, t, T_1, \ldots) \, e^{-y}, \quad \tilde{h}_1 = \tilde{h}_{11}(x, y, t) + C_h^{(1)}(x, t, T_1, \ldots) \, e^{-y}. \tag{5.19a,b}$$

Here $C_u^{(1)}$ and $C_h^{(1)}$ are arbitrary functions which have to be determined and

$$\tilde{u}_{11} = -\frac{1}{2} \left(e^y \int_y^\infty \tilde{F}_u^{(1)} \, e^{-y} \, \mathrm{d}y + e^{-y} \int_0^y \tilde{F}_u^{(1)} \, e^y \, \mathrm{d}y \right), \tag{5.20a}$$

$$\tilde{h}_{11} = -\frac{1}{2} \left(e^y \int_y^\infty \tilde{F}_h^{(1)} \, e^{-y} \, \mathrm{d}y + e^{-y} \int_0^y \tilde{F}_h^{(1)} \, e^y \, \mathrm{d}y \right). \tag{5.20b}$$

To determine $C_u^{(1)}$ and $C_h^{(1)}$ we write (5.1*a,b,c*) at $y = 0$ (cf. the derivation of (3.22)):

$$\frac{\partial \tilde{u}_1}{\partial t} + \frac{\partial \tilde{h}_1}{\partial x} = \tilde{R}_u^{(1)}|_{y=0}, \quad \tilde{u}_1 = -\frac{\partial \tilde{h}_1}{\partial y}, \quad \tilde{h}_1 = -\frac{\partial \tilde{u}_1}{\partial y} - \tilde{R}_\zeta^{(1)}|_{y=0} \quad \text{at} \quad y = 0. \tag{5.21a–c}$$

By virtue of (5.19) we have

$$(\tilde{u}_1, \tilde{h}_1)_{y=0} = -\frac{1}{2} \int_0^\infty (\tilde{F}_u^{(1)}, \tilde{F}_h^{(1)}) \, e^{-y} \, dy + (C_u^{(1)}, C_h^{(1)}), \qquad (5.22a)$$

$$\left(\frac{\partial \tilde{u}_1}{\partial y}, \frac{\partial \tilde{h}}{\partial y} \right)_{y=0} = -\frac{1}{2} \int_0^\infty (\tilde{F}_u^{(1)}, \tilde{F}_h^{(1)}) \, e^{-y} \, dy - (C_u^{(1)}, C_h^{(1)}). \qquad (5.22b)$$

Substitution of (5.22) into (5.21b, c) gives

$$C_u^{(1)} - C_h^{(1)} = \frac{1}{2} \int_0^\infty (\tilde{F}_u^{(1)} + \tilde{F}_h^{(1)}) \, e^{-y} \, dy, \qquad (5.23a)$$

$$C_u^{(1)} - C_h^{(1)} = -\frac{1}{2} \int_0^\infty (\tilde{F}_u^{(1)} + \tilde{F}_h^{(1)}) \, e^{-y} \, dy + \tilde{R}_\zeta^{(1)}|_{y=0}. \qquad (5.23b)$$

Equations (5.23) are compatible since

$$\int_0^\infty (\tilde{F}_u^{(1)} + \tilde{F}_h^{(1)}) \, e^{-y} \, dy = \tilde{R}_\zeta^{(1)}|_{y=0}. \qquad (5.24)$$

The validity of (5.24) can be shown using the identity

$$\tilde{F}_u^{(1)} + \tilde{F}_h^{(1)} = \frac{\partial F}{\partial y} - F, \quad F = \frac{\partial \tilde{v}_1}{\partial x} - \frac{\partial \tilde{v}_1}{\partial t} + \tilde{R}_v^{(1)} - \tilde{R}_\zeta^{(1)}. \qquad (5.25a, b)$$

Thus we have

$$C_u^{(1)} = C_h^{(1)} + \tfrac{1}{2} \tilde{R}_\zeta^{(1)} \big|_{y=0}. \qquad (5.26)$$

Substituting (5.22a), (5.26) into (5.21a) we obtain the equation for $C_h^{(1)}$:

$$\frac{\partial C_h^{(1)}}{\partial t} + \frac{\partial C_h^{(1)}}{\partial x} = \frac{1}{2} \int_0^\infty \left(\frac{\partial \tilde{F}_u^{(1)}}{\partial t} + \frac{\partial \tilde{F}_h^{(1)}}{\partial t} \right) e^{-y} \, dy + \tilde{R}_u^{(1)}|_{y=0} - \frac{1}{2} \frac{\partial \tilde{R}_\zeta^{(1)}}{\partial t} \bigg|_{y=0}. \qquad (5.27)$$

Using the formula

$$\frac{\partial \tilde{F}_u^{(1)}}{\partial t} + \frac{\partial \tilde{F}_h^{(1)}}{\partial x} = \frac{\partial^2 \tilde{v}_1}{\partial t^2} - \frac{\partial^2 \tilde{v}_1}{\partial x^2} + \frac{\partial^2 \tilde{R}_v^{(1)}}{\partial x \, \partial y} - \frac{\partial \tilde{R}_v^{(1)}}{\partial t} + \frac{\partial \tilde{R}_\zeta^{(1)}}{\partial x} - \frac{\partial^2 \tilde{R}_\zeta^{(1)}}{\partial t \partial y} \qquad (5.28)$$

and integration by parts we write (5.27) in the form

$$\frac{\partial C_h^{(1)}}{\partial t} + \frac{\partial C_h^{(1)}}{\partial x} = R_k = \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x} \right)$$

$$\times \left[\frac{1}{2} \int_0^\infty \left(\frac{\partial \tilde{v}_1}{\partial t} - \frac{\partial \tilde{v}_1}{\partial x} + \tilde{R}_v^{(1)} + \tilde{R}_\zeta^{(1)} \right) e^{-y} \, dy - \left(\frac{\tilde{u}_0^2}{2} + \bar{u}_0 \tilde{u}_0 \right)_{y=0} \right]$$

$$- \frac{\partial \tilde{u}_0}{\partial T_1} \bigg|_{y=0} + \frac{\partial}{\partial t} \left[\left(\frac{\tilde{u}_0^2}{2} + \bar{u}_0 \tilde{u}_0 \right)_{y=0} - \int_0^\infty (\tilde{R}_v^{(1)} + \tilde{R}_\zeta^{(1)}) \, e^{-y} \, dy \right]. \qquad (5.29)$$

Equation (5.29) describes the first-order Kelvin waves plus some function bounded in time and space. Analysis of resonant terms on the right-hand side R_k of (5.29) gives us the equation for slow evolution of the lowest-order Kelvin wave.

5.5. *Slow evolution of the lowest-order Kelvin wave*

The term $(\partial/\partial t + \partial/\partial x)[\ldots]$ in R_k is non-resonant while the terms $(\partial/\partial t)(\frac{1}{2}\tilde{u}_0^2)_{y=0}$ and $\partial \tilde{u}_0/\partial T_1|_{y=0}$ are definitely resonant, so the question is whether the other terms in R_k are resonant. To answer this the response to various nonlinear terms on the right-hand side of (5.29) was examined in an analogous way to the nonlinear source terms in (5.13), (5.14). for the localized initial conditions (2.9c) only the interactions Kelvin wave–Kelvin wave are resonant and the other interactions (Kelvin wave–IG waves, Kelvin wave–slow component, IG waves–IG waves, and IG waves–slow component) are not resonant. Therefore the resulting equation for the slow evolution of the lowest-order Kelvin wave in the localized case has the form

$$\frac{\partial K_w^{(0)}}{\partial T_1} + K_w^{(0)}\frac{\partial K_w^{(0)}}{\partial x'} = 0, \quad x' = x - t. \tag{5.30}$$

Equation (5.30) should be solved under initial condition (3.34a).

In the periodic and 'step' cases the interaction Kelvin wave–slow component turns out to be resonant in addition to the Kelvin wave–Kelvin wave interactions; all other interactions are ineffective. The term in R_k related to the Kelvin wave–slow component interaction using (5.1h) and (3.16a) can be written in the form

$$R_{km} = \int_0^\infty \left(\frac{\partial K_w^{(0)}}{\partial t}\frac{\partial \bar{v}_0}{\partial x} + K_w^{(0)}\frac{\partial \Pi_0}{\partial x} + \frac{\partial K_w^{(0)}}{\partial x}\Pi_0 \right) e^{-2y}\,dy + \left(\bar{u}_0\frac{\partial K_w^{(0)}}{\partial t} \right)_{y=0}. \tag{5.31}$$

Each term in the integrand and in the brackets on the right-hand side of (5.31) can be represented as a product:

$$P = a_k(x, y, t, T_1, \ldots)b_s(x, y, T_1, \ldots), \tag{5.32}$$

where a_k and b_s correspond to the Kelvin wave and slow component, respectively. In the periodic case both these terms are represented as Fourier series in x:

$$a_k = e^{-y}\sum_m a_{km}(T_1, \ldots)\,e^{-im(x-t)}, \quad b_s = \sum_n b_{sn}(y, T_1, \ldots)\,e^{-imx}, \tag{5.33a,b}$$

and therefore (5.31) takes the form

$$R_{km} = \sum_{m,n} \bar{a}_{km}\bar{b}_{sn}\,e^{i(m+n)x-ixt} \tag{5.34}$$

where $\bar{a}_{km}$ and $\bar{b}_{sn}$ are coefficients depending on slow times. Clearly, the sum (5.34) is resonant only if the coefficient $\bar{b}_{s0}$ and therefore b_{s0} are non-zero, i.e. the slow component includes a zonal current. Simple calculations using (3.6a, b, c) show that the resonant part of R_{km} can be written as

$$R_{km}^{(r)} = -C(T_1, \ldots)\frac{\partial K_w^{(0)}}{\partial x}, \tag{5.35}$$

where

$$C = -3\int_0^\infty \bar{h}_{00}\,e^{-2y}\,dy + 2\bar{h}_{0B}, \quad \bar{h}_{00}(y, T_1, \ldots) = \langle \bar{h}_0 \rangle_x, \quad \bar{h}_{0B} = \bar{h}_0|_{y=0}. \tag{5.36a,b,c}$$

The resulting equation for the slow evolution of the lowest-order Kelvin wave is written as

$$\frac{\partial K_w^{(0)}}{\partial T_1} + [K_w^{(0)} + C(T_1, \ldots)]\frac{\partial K_w^{(0)}}{\partial x'} = 0. \tag{5.37}$$

The initial condition for (5.37) is given by equation (3.32a).

In the step case (2.9b) the resonant part of R_{km} has a form similar to (5.35) but with the coefficient depending on x:

$$R_{km}^{(r)} = -C_{st}(x, T_1, \ldots) \frac{\partial K_w^{(0)}}{\partial x}, \quad C_{st} = -3 \int_0^\infty \bar{h}_0 \, \mathrm{e}^{-2y} \, \mathrm{d}y + 2\bar{h}_{0B}. \tag{5.38a, b}$$

The x-dependence is related to the integral $F = \int_0^\infty \bar{h}_0 \, \mathrm{e}^{-2y} \, \mathrm{d}y$ which has the form of a step (see above, §4):

$$F = F(x, T_1, \ldots) \to F^{(\pm)}, \quad x \to \pm\infty, \tag{5.39}$$

$$F^{(\pm)} = \int_0^\infty \bar{h}_0^{(\pm)}(y) \, \mathrm{e}^{-2y} \, \mathrm{d}y, \quad \bar{h}_0^{(\pm)} = \lim \bar{h}_0, \quad x \to \pm\infty. \tag{5.40}$$

We demand that the solution be bounded as $t \to \infty$ in any coordinate frame including one moving with the Kelvin-wave phase speed. It follows from this condition that the resonant part $R_{km}^{(r)}$ takes the form (5.38a) but with constant parameter C equal to

$$C = -3 \int_0^\infty \bar{h}_0^{(+)} \, \mathrm{e}^{-2y} \mathrm{d}y + 2\bar{h}_{0B}. \tag{5.41}$$

Correspondingly, the equation for slow evolution of the Kelvin wave in the step case has the form (5.37) but with constant C given by (5.41). Note that (5.37) can be reduced to (5.30) by the simple coordinate transformation

$$x' \to x' - X_C(T_1, \ldots), \quad \frac{\partial X_C}{\partial T_1} = C. \tag{5.42a, b}$$

Thus we see that in all cases the slow evolution of Kelvin waves is governed by the equation for a simple wave (5.30); the presence of mean zonal current in the periodic and step cases results only in a Doppler shift of the Kelvin-wave phase speed. It is well-known (e.g. Lighthill 1980) that in general the simple wave breaks in a finite time, and therefore the Kelvin wave behaviour can be characterized as fast propagation of a slowly breaking profile. Note that the breaking can be prevented if some additional dispersion or friction is incorporated in the model.

5.6. *Mass conservation in the localized case*

In §4.3 we showed that in the localized case the total mass of the slow component was not conserved, and the first-order slow correction was not localized. Below, we will see that the first-order Kelvin wave is also non-localized, and together with the non-localized first-order slow correction compensates the non-conservation of the lowest-order slow field mass.

The first-order correction $\bar{h}_1$ obeys the equation following from (4.4a, b, c):

$$\nabla^2 \bar{h}_1 - \bar{h}_1 = \Pi_1 + \mathrm{div}\bar{R}_v^{(1)}, \tag{5.43}$$

where $\bar{R}_v^{(1)}$ is given by (5.5b). To obtain the boundary condition for (5.43) we use equation (4.6) in a slightly modified form,

$$\left.\frac{\partial \bar{h}_1}{\partial x}\right|_{y=0} = \left.\frac{\partial^2 \bar{h}_0}{\partial y \, \partial T_1}\right|_{y=0} - \left.\bar{u}_0 \frac{\partial \bar{u}_0}{\partial x}\right|_{y=0}, \tag{5.44}$$

one of the conditions (5.22a),

$$\bar{h}|_{y=0} = -\frac{1}{2} \int_0^\infty \tilde{F}_h^{(1)} \, \mathrm{e}^{-y} \, \mathrm{d}y + C_h^{(1)}, \tag{5.45}$$

and the presumed locality of the solution to problem (2.1a, b), (2.3), (2.5) with localized initial conditions (2.9c).

The solution $C_h^{(1)}$ to (5.29) can be represented (since all nonlinear terms are localized and tend to zero at $t \to \infty$) as a sum:

$$C_h^{(1)} = K_w^{(1)}(x - t, T_1, \ldots) + C_L(x, t),\tag{5.46}$$

where $K_w^{(1)}$ is the first-order Kelvin wave amplitude and C_L a localized solution to the inhomogeneous equation (5.27). Since we are looking for a localized solution, the sum $h_1 = \bar{h}_1 + \tilde{h}_1$ should be space-localized. Therefore, using (5.36) and the fact that the function $\tilde{F}_h^{(1)}$ is localized, (5.45) can be written as

$$\bar{h}_1|_{y=0} = \bar{h}_{1B}(x, T_1, \ldots) = -K_w^{(1)}(x - t, T_1, \ldots) + F_L,\tag{5.47}$$

where F_L is a localized function. The Kelvin wave $K_w^{(1)}(x - t, T_1, \ldots)\,\mathrm{e}^{-y}$ should have zero time average (see (3.28)), therefore from (5.47)

$$\bar{h}_{1B} \to 0, \quad x \to -\infty.\tag{5.48}$$

The boundary condition for $\bar{h}_1$ follows from (5.44), (5.48):

$$\bar{h}_1|_{y=0} = \int_{-\infty}^{x} \left.\frac{\partial^2 \bar{h}_0}{\partial y\, \partial T_1}\right|_{y=0} \mathrm{d}x - \left.\frac{\bar{u}_0^2}{2}\right|_{y=0}.\tag{5.49}$$

The solution $\bar{h}_1$ to the problem (5.43), (5.49) can be represented as a sum of non-localized and localized terms, the non-localized term arising when the along-wall circulation

$$\int_{-\infty}^{\infty} \left.\frac{\partial^2 \bar{h}_0}{\partial y\, \partial T_1}\right|_{y=0} \mathrm{d}x$$

is non-zero. The function Π_1 in (5.43) is localized since by virtue of (4.4c), (5.1c) $\zeta_1 - h_1 = \Pi_1 + \tilde{R}_\zeta^{(1)}$ and $\tilde{R}_\zeta^{(1)}$ is localized. Therefore $\bar{h}_1$ is represented in the form

$$\bar{h}_1 = \bar{h}_{1B}(x, T_1, \ldots)\,\mathrm{e}^{-y} + \bar{h}_{11},\tag{5.50}$$

where $\bar{h}_{11}$ obeys

$$\nabla^2 \bar{h}_{11} - \bar{h}_{11} = \Pi_1 + \mathrm{div}\,\overline{\boldsymbol{R}}_v^{(1)} - \frac{\partial^2 \bar{h}_{1B}}{\partial x^2}\,\mathrm{e}^{-y}, \quad \bar{h}_{11}|_{y=0} = 0.\tag{5.51a, b}$$

The right-hand side of (5.51a) is localized, and therefore $\bar{h}_{11}$ is also localized.

We now write the elevation h up to $O(\varepsilon^2)$ as

$$h = \bar{h}_0 + \tilde{h}_{01} + K_w^{(0)}(x - t, T_1, \ldots)\,\mathrm{e}^{-y}$$
$$+ \varepsilon[\bar{h}_1 + \tilde{h}_{11} + C_L\,\mathrm{e}^{-y} + K_w^{(1)}(x - t, T_1, \ldots)\,\mathrm{e}^{-y}],\tag{5.52}$$

and integrate (5.52) over the half-plane $y > 0$. It readily follows from (3.17b, c) that the 'mass' of the fast field $\tilde{h}_{01}$ is conserved (taking into account that $\tilde{v}_0$ in (3.17c) is odd in y):

$$\int_{y>0} \bar{h}_{01}\,\mathrm{d}x\,\mathrm{d}y = \text{constant}.\tag{5.53}$$

This is also valid for the lowest-order Kelvin wave solution,

$$\int_{y>0} K_w^{(0)}(x - t, T_1, \ldots)\,\mathrm{e}^{-y}\,\mathrm{d}x\,\mathrm{d}y = \text{constant},\tag{5.54}$$

as follows from (5.30). Also, the calculations show that the changes of the 'mass' of the fast field $\tilde{h}_{11}$ are compensated by the term $C_L e^{-y}$, i.e.

$$\frac{\partial}{\partial t} \int_{y>0} [\tilde{h}_{11} + C_L e^{-y}] dx\, dy = 0. \tag{5.55}$$

To obtain (5.55) we use equations (5.20*b*), (5.24), and the equation for C_L, which coincides with (5.27) where $C_h^{(1)}$ is replaced by C_L.

So, for the total mass to be conserved the term

$$S_0 - \frac{\partial}{\partial T_1} \int_{y>0} \bar{h}_0\, dx\, dy = - \int \frac{\partial^2 \bar{h}_0}{\partial y\, \partial T_1}\bigg|_{y=0} dx \tag{5.56}$$

should be balanced by the term

$$S_1 = \frac{\partial}{\partial t} \int_{y>0} [\bar{h}_1 + K_w^{(1)}(x - t, T_1, \ldots) e^{-y}] dx\, dy. \tag{5.57}$$

To show that this is really the case we note that by virtue of (5.47), (5.49)

$$K_w^{(1)} = - \int_{-\infty}^{\infty} \frac{\partial^2 \bar{h}_0}{\partial y\, \partial T_1}\bigg|_{y=0} dx, \quad x \to \infty, \tag{5.58}$$

whence it follows that the first-order Kelvin wave is not localized, it has a step-like shape if

$$\int_{-\infty}^{\infty} \frac{\partial^2 \bar{h}_0}{\partial y\, \partial T_1}\bigg|_{y=0} dx \approx 0.$$

Nevertheless the integral in (5.57) converges since the sum $\bar{h}_1 + K_w^{(1)}(x-t, T_1, \ldots) e^{-y}$ is localized as follows from (5.58), (5.50), and (5.47):

$$\bar{h}_1 + K_w^{(1)}(x - t, T_1, \ldots) e^{-y} = [\bar{h}_{1B} + K_w^{(1)}(x - t, T_1, \ldots)] e^{-y} + \bar{h}_{11}. \tag{5.59}$$

Using (5.59), (5.47), and (5.58) one can calculate S_1:

$$S_1 = \int_{-\infty}^{\infty} \frac{\partial F_L}{\partial t} dx = - \int_{-\infty}^{\infty} \frac{\partial K_w^{(1)}}{\partial t} dx = - \int_{-\infty}^{\infty} \frac{\partial^2 \bar{h}_0}{\partial y\, \partial T_1}\bigg|_{y=0} dx. \tag{5.60}$$

Thus we see that the terms S_0 and S_1 do balance each other and the total mass (fast + slow) is conserved. The sum $\bar{h}_1 + K_w^{(1)}(x - t, T_1, \ldots) e^{-y}$ is reminiscent of an injected jet bringing out the surplus or shortage (depending on the sign of $\int_{-\infty}^{\infty} \partial^2 \bar{h}_0/\partial y \partial T_1|_{y=0}\, dx$) of the mass from the localized lowest-order disturbance.

5.7. *Initial slow field $\bar{h}_{1I}$*

The initial slow field $\bar{h}_{1I}$ obeys the equation (see (5.43))

$$\nabla^2 \bar{h}_{1I} - \bar{h}_{1I} = \Pi_{1I} + \text{div}\overline{\boldsymbol{R}}^{(1)\text{rl}}. \tag{5.61}$$

To obtain the boundary condition for (5.61) at $y = 0$ we use equation (5.45) analogously to (3.20*a*). Since $\tilde{h}_{1I} = -\bar{h}_{1I}$ we have from (5.45):

$$\bar{h}_{1I}|_{y=0} = \frac{1}{2} \int_0^\infty \tilde{F}_{hI}^{(1)} e^{-y} dy - K_w^{(1)}(x) - C_{hfI}^{(1)}, \tag{5.62}$$

where the function $\tilde{F}_{hI}^{(1)}$ (see (5.18*b*)) is known because $\tilde{v}_{1I}$ and $\partial \tilde{v}_1/\partial t|_{t=0}$ are given by (5.10), (5.11). The function $C_h^{(1)}$ in (5.45) is replaced by the sum $C_h^{(1)} = K_w^{(1)}$

$(x - t, T_1, \ldots) + C_{hf}^{(1)}$, where $K_w^{(1)}$ is the amplitude of the first-order Kelvin wave, and $C_{hf}^{(1)}$ is a known particular solution of the inhomogeneous equation (5.27). Again, applying the condition (3.28) to $K_w^{(1)}$ one can determine either $\bar{h}_{1I}|_{x=-\infty, y=0}$ in the step case of $\langle \bar{h}_{1I}|_{y=0} \rangle_x$ in the periodic case, and then $\bar{h}_{1I}|_{y=0}$ from equation (4.6) taken at the initial moment. Given $\bar{h}_{1I}|_{y=0}$ we can find from (5.62) the initial profile $K_w^{(1)}(x)$ of the first-order Kelvin wave, and the field $\bar{h}_{1I}$ from (5.61).

Thus the first-order splitting (4.2) is self-consistent for all initial conditions (2.9). The algorithm in use provides the evolution equation for the lowest-order slow field and the Kelvin waves, and a fast correction to the lowest-order fast component. The evolution of the first correction to the slow field is determined at the third order of the perturbation theory (cf. RZB).

6. Modified QG equation

The equation for the first-order slow motion is obtained in a manner similar to the method used in § 3.3 of RZB. We write the third-order vorticity equation as

$$\frac{\partial(\zeta_2 - h_2)}{\partial t} + \frac{\partial(\zeta_1 - h_1)}{\partial T_1} + \frac{\partial \Pi_0}{\partial T_2} + \frac{\partial[u_0(\zeta_1 - h_1)]}{\partial x}$$

$$+ \frac{\partial[v_0(\zeta_1 - h_1)]}{\partial y} + \frac{\partial(u_1 \Pi_0)}{\partial x} + \frac{\partial(v_1 \Pi_0)}{\partial y} = 0, \quad (6.1)$$

and average it in time applying (3.5). Using representations (3.3), (4.2) and properties of the fast fields described in §§ 3, 5 one arrives at

$$\frac{\partial \Pi_1}{\partial T_1} + \frac{\partial \Pi_0}{\partial T_2} + \frac{\partial(\bar{u}_0 \Pi_1)}{\partial x} + \frac{\partial(\bar{v}_0 \Pi_1)}{\partial y} + \frac{\partial(\bar{u}_1 \Pi_0)}{\partial x} + \frac{\partial(\bar{v}_1 \Pi_0)}{\partial y} + \frac{\partial \langle \tilde{u}_0(\tilde{\zeta}_1 - \tilde{h}_1) \rangle_t}{\partial x} = 0. \quad (6.2)$$

By virtue of (5.1c, g) the average $\langle \tilde{u}_0(\tilde{\zeta}_1 - \tilde{h}_1) \rangle_t$ is represented as

$$\langle \tilde{u}_0(\tilde{\zeta}_1 - \tilde{h}_1) \rangle_t = \langle \tilde{u}_0 \tilde{R}_s^{(1)} \rangle_t = -\left\langle \tilde{u}_0 \frac{\partial \tilde{U}_0}{\partial x} \right\rangle_t \Pi_0. \quad (6.3)$$

Since the fields $\tilde{u}_{01}$, $\tilde{U}_{01}$ in (3.16a), (5.4a) consist of IG waves, decaying with increasing time, equation (6.3) is reduced to the equation (see (3.16a), (3.23), and (5.4a))

$$\langle \tilde{u}_0(\tilde{\zeta}_1 - \tilde{h}_1) \rangle_t = C_K \, e^{-2y} \Pi_0, \quad C_K = \langle [K_w^{(0)}(x - t, T_1, \ldots)]^2 \rangle_t. \quad (6.4a, b)$$

The coefficient C_K is zero for the step and localized initial conditions because in these cases the Kelvin wave amplitude $K_w^{(0)}(x - t, T_1, \ldots)$ tends to zero as $x \to -\infty$ (see (3.28)). But in the periodic case the averaging (3.5) gives

$$C_K = \frac{1}{2\pi} \int_0^{2\pi} [K_w^{(0)}(z, T_1, \ldots)]^2 \, dz. \quad (6.5)$$

As readily follows from (5.37) the coefficient C_K is conserved in time (at least, for the time T_1), and therefore it can be calculated directly from the initial Kelvin profile (3.32a).

In other to obtain a closed equation for $\bar{h}_0$, $\bar{h}_1$ we use (4.4a, b, e, f) to express Π_1 as

$$\Pi_1 = \frac{\partial \bar{v}_1}{\partial x} - \frac{\partial \bar{u}_1}{\partial y} - \bar{h}_1 = \nabla^2 \bar{h}_1 - \bar{h}_1 - 2J\left(\frac{\partial \bar{h}_0}{\partial x}, \frac{\partial \bar{h}_0}{\partial y} \right). \quad (6.6)$$

With the help of (4.4a,b,e,f) and the evolution (4.3) for $\Pi_0 = \nabla^2 \bar{h}_0 - \bar{h}_0$, equation (6.2) takes the following form:

$$\frac{\partial \Pi_0}{\partial T_2} + \frac{\partial}{\partial T_1}(\Pi_1 - \nabla \bar{h}_0 \nabla \Pi_0 - \bar{h}_0 \Pi_0) + J(\bar{h}_0, \Pi_1 - \nabla \bar{h}_0 \nabla \Pi_0 - \bar{h}_0 \Pi_0)$$

$$+ J\left(\bar{h}_1 - \frac{(\nabla \bar{h}_0)^2}{2} + \frac{C_K}{2} e^{-2y}, \Pi_0\right) = 0. \quad (6.7)$$

Equation (6.7) describes a next-order correction to (4.3), which is necessary to take into account the slow evolution of the balanced component of the flow for times much longer than T_1. One can combine the two equations by introducing a 'full' slow elevation $\bar{h} = \bar{h}_0 + \varepsilon \bar{h}_1$ (cf. RZB). The resulting equation (which will be referred to as the modified quasi-geostrophic potential vorticity (MQGPV) equation) can be written in the form of conservation of potential vorticity Π_M:

$$\frac{\partial \Pi_M}{\partial T_1} + J\left(\bar{h} - \varepsilon \frac{(\nabla \bar{h})^2}{2} + \varepsilon \frac{C_K}{2} e^{-2y}, \Pi_M\right) = 0, \quad (6.8a)$$

$$\Pi_M = \nabla^2 \bar{h} - \bar{h} - \varepsilon\left[\bar{h}(\nabla^2 \bar{h} - \bar{h}) + \nabla \bar{h} \nabla(\nabla^2 \bar{h} - \bar{h}) + 2J\left(\frac{\partial \bar{h}}{\partial x}, \frac{\partial \bar{h}}{\partial y}\right)\right]. \quad (6.8b)$$

Equation (6.8) is constructed in such a way that if the solution $\bar{h}$ is represented as an asymptotic multiple time-scales series,

$$\bar{h} = \bar{h}_0(x, y, T_1, \ldots) + \varepsilon \bar{h}_1(x, y, T_1, \ldots) + \cdots, \quad (6.9)$$

then the equations for the corrections $\bar{h}_0$, $\bar{h}_1$ coincide with (4.3) and (6.7), respectively. Obviously, (6.8) is valid up to terms $O(\varepsilon^2)$.

In the step and localized cases the coefficient C_K is zero, and MQGPV equation (6.8) coincides with the 'improved' QGPV equation derived in RZB. At the same time, in the periodic case $C_K \neq 0$ and the Kelvin wave makes a direct contribution to the slow dynamics, the contribution depending on the initial conditions (2.2). This means, generally, that in the presence of a boundary MQGPV equation (6.8) cannot be correctly obtained by a direct expansion of (2.1a,b), (2.5) in ε assuming all variables to depend on slow times only, as occurs in the case of an unbounded region (cf. RZB).

The initial conditions for (6.8) are determined by the problem (3.35), (3.29) for $\bar{h}_{0I}$ and equation (5.61) with the corresponding boundary condition for $\bar{h}_{1I}$ (see §5.7). The boundary conditions for (6.8) follow from (4.5), (4.6), and from the condition analogous to (4.6) for the second-order slow field $\bar{h}_2$,

$$\left.\frac{\partial \bar{h}_2}{\partial x}\right|_{y=0} = -\left(\frac{\partial \bar{u}_1}{\partial T_1} + \frac{\partial \bar{u}_0}{\partial T_2} + \frac{\partial}{\partial x}(\bar{u}_0 \bar{u}_1)\right)_{y=0}, \quad (6.10)$$

which is derived in the same way as (4.6). In the periodic and step cases we have from (4.6)

$$\bar{h}_1|_{y=0} = \int \left.\frac{\partial^2 \bar{h}_0}{\partial y \, \partial T_1}\right|_{y=0} dx - \left.\frac{\bar{u}_0^2}{2}\right|_{y=0} + \bar{h}_{1B}(T_1, \ldots), \quad (6.11)$$

where the first term on the right-hand side of (6.11) is the anti-derivative of $\partial^2 \bar{h}_0 / \partial y \, \partial T_1|_{y=0}$ and $\bar{h}_{1B}$ is obtained from the condition of boundedness of $\bar{h}_2$

using (6.10) and (4.4*b*):

$$\left\langle \left(\frac{\partial \bar{u}_1}{\partial T_1} + \frac{\partial \bar{u}_0}{\partial T_2} \right)_{y=0} \right\rangle_x = -\left\langle \left(\frac{\partial^2 \bar{h}_1}{\partial y \, \partial T_1} + \frac{\partial^2 \bar{h}_0}{\partial y \partial T_2} \right)_{y=0} \right\rangle_x = 0. \qquad (6.12)$$

Combining (4.5) with (6.11), and (4.7) with (6.12) we obtain the boundary conditions

$$\left[\bar{h} - \varepsilon \int \frac{\partial^2 \bar{h}}{\partial y \partial T_1} dx + \frac{\varepsilon}{2} \left(\frac{\partial \bar{h}}{\partial y} \right)^2 \right]_{y=0} = \bar{h}_B(T_1, \ldots), \qquad (6.13a)$$

$$\left\langle \frac{\partial^2 \bar{h}}{\partial y \, \partial T_1} \bigg|_{y=0} \right\rangle_x = 0. \qquad (6.13b)$$

Here (6.13*b*) serves to determine the unknown function $\bar{h}_B$ in (6.13*a*). In the localized case (6.13*b*) is satisfied identically and $\bar{h}_B = 0$ in (6.13*a*).

The MQGPV equation (6.8) together with the initial condition $\bar{h}_I = \bar{h}_{0I} + \varepsilon \bar{h}_{1I}$ and the boundary conditions (6.13) allows us to determine the evolution of the slow component up to non-dimensional times $O(\varepsilon^{-2})$.

7. Summary and conclusion

We have described the process of nonlinear geostrophic adjustment in the barotropic RSW model on a half-plane bounded by a rigid wall, applying the multiple-time-scale perturbation theory developed in RZB and based on the assumption of a small Rossby number. Different initial states (all localized in the y – perpendicular to the wall – direction) were considered: periodic in x, 'step'-like (tending to along-wall rectilinear flows as $x \to \pm\infty$), and localized ones.

Generally, the geostrophic adjustment considered is similar to that in the unbounded plane examined in RZB. In all cases the initial perturbation is split in a unique way into slow and fast components evolving with characteristic time scales f^{-1} and $(\varepsilon f)^{-1}$, respectively. The slow component is not influenced by the fast one, at least for times $t \leqslant (\varepsilon f)^{-1}$, and remains close to geostrophic balance. The fast component consists mainly of linear IG waves rapidly propagating outward from the initial disturbance and Kelvin waves confined near the boundary. Like the unbounded case considered in RZB, the nonlinear interactions of IG waves with each other, with the slow component and with the Kelvin waves result in only a small correction to the fast field.

At the same time, the presence of the boundary brings a number of new effects into the geostrophic adjustment process. First, Kelvin waves arise which propagate along the boundary in such a way that the boundary is on the right in the direction of propagation. The theory provides simple formulae allowing us to construct the initial profile of the Kelvin wave given arbitrary initial conditions. The lowest-order fields of the theory describe the process of linear geostrophic adjustment of an arbitrary initial state on the bounded half-plane. With increasing time the Kelvin wave profile slowly distorts, the distortion being due to the Kelvin wave nonlinear self-interaction and being described by the simple-wave equation. The resulting evolution of the initial Kelvin wave profile can be characterized as fast propagation, with slow breaking.

Probably the most important result of this work is that the presence of Kelvin waves does not prevent the fast–slow splitting in spite of the fact that (unlike IG waves, see RZB) the frequency gap between the Kelvin waves and slow motion is

absent (i.e. Kelvin waves with arbitrarily small frequencies can exist). The possibility of splitting is explained by the special structure of the Kelvin waves in each case considered here. For the periodic initial conditions the Kelvin wave profile is also periodic but it has zero mean, which physically corresponds to the absence of a zero-frequency harmonic in the Kelvin wave spectrum. For the localized and step-like initial conditions such harmonics can be present in the Kelvin wave spectrum, but the shape of its profile is such that the Kelvin wave field rapidly decays at a fixed spatial point when the Kelvin wave propagates off this point. One can assume that the Kelvin wave breaking does not affect the slow mode in the localized and step-like cases since it happens far from the main regions of 'slow variability'. For the periodic motion the effect of the Kelvin wave breaking on the geostrophic mode should be examined more thoroughly. An important role in this case can be played by the non-conservation of potential vorticity across a Kelvin shock, revealed in numerical experiments by Helfrich, Kuo & Pratt (1999).

Evolution of the slow motion on times $t \leqslant (\varepsilon f)^{-1}$ is governed by the well-known quasi-geostrophic potential vorticity equation for the elevation $\bar{h}_0$. The theory provides an algorithm to determine the initial slow and fast fields to any order in ε. The boundary condition used in this non-viscous model is the no-flux condition on the rigid wall. Being formulated in terms of $\bar{h}_0$, this condition contains an arbitrary function $\bar{h}_{0B}(T_1, \ldots)$ which is constant along the boundary but depends on the slow time (e.g. Kamenkovich & Reznik 1978; Pedlosky 1987).

To remove this uncertainty the conservation of along-boundary circulation is used (e.g. Pedlosky 1987). If the fast component is absent then the slow circulation is assumed to be conserved, this conservation providing the energy and mass conservation of the lowest-order slow motion. However, if the initial conditions are not balanced, so that the fast component is present, then the *total* circulation (fast + slow) should be conserved. The question is whether the fast and slow circulations are conserved separately. Our analysis demonstrates that this is the case for the periodic and step-like initial conditions.

The situation for localized initial conditions is somewhat more complicated. The lowest-order slow motion in this case is also localized and the elevation $\bar{h}_0$ is zero at the boundary, i.e. the problem for $\bar{h}_0$ is well-defined without using the conservation of slow circulation. Moreover, a simple analysis shows that, generally, the lowest-order circulation and, therefore, the total mass of slow localized motion are not conserved. Conservation of the total circulation and mass is provided by the first-order slow correction $\bar{h}_1$ and Kelvin wave $K_w^{(1)} e^{-y}$. Separately, neither $\bar{h}_1$, nor the Kelvin wave amplitude $K_w^{(1)}$, are localized in x but the sum $\bar{h}_1 + K_w^{(1)} e^{-y}$ is localized and is reminiscent of an injected jet carrying the surplus, or shortage, of mass from the localized lowest-order slow disturbance.

Note that Dorofeyev & Larichev (1992) met an analogous problem when considering the reflection of linear Rossby waves from the meridional boundary in the framework of a shallow water model on the β-plane. They revealed that the total mass of Rossby waves is not conserved, and the surplus or shortage of mass is carried by fast Kelvin waves. Also, Helfrich & Pedlosky (1995) examined the QG motion in periodic and unbounded channels and indicated that in the localized case non-conservation of slow circulation results in radiation of Kelvin waves 'emanating from the local region' of slow motion.

On longer times $t \leqslant (\varepsilon^2 f)^{-1}$ the slow motion obeys the so-called modified quasi-geostrophic potential vorticity (MQGPV) equation. The theory provides initial and boundary conditions for this equation. The MQGPV equation coincides exactly with

the 'improved' QGPV equation, derived in RZB, in the step-like and localized cases. In the periodic case this equation contains an additional term due to the Kelvin-wave self-interaction, this term depending on the initial Kelvin wave profile. This means, generally, that in the presence of a boundary the MQGPV equation cannot be correctly obtained by a direct expansion of the RSW system in ε assuming all variables to depend on slow times only, as occurs in the case of unbounded region.

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